

ON LINEAR OPERATORS EXTENDING [PSEUDO]METRICS

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ABSTRACT. For every closed subset X of a stratifiable [resp. metrizable] space Y we construct a positive linear extension operator $T : \mathbb{R}^{X \times X} \rightarrow \mathbb{R}^{Y \times Y}$ preserving constant functions, bounded functions, continuous functions, pseudometrics, metrics, [resp. dominating metrics, and admissible metrics]. This operator is continuous with respect to each of the three topologies: point-wise convergence, uniform, and compact-open.

An equivariant analog of the above statement is proved as well.

The problem of existing a linear operator extending [pseudo]metrics from a closed subset of a metric compactum X over all of X was posed by the second author in [4] and partly solved in [4], [5]. A complete solution of this problem appeared in [2] and [15] (see also [1] and [3]). M. Zarichnyi [16] presented a very simple construction of such extension operators.

In contrast to the mentioned results, the present paper, which is a simplified and generalized version of the preprint [1], allows to construct linear operators extending metrics which are continuous with respect to the point-wise convergence of functions.

For a space Z we denote by \mathbb{R}^Z the space of all, not necessarily continuous, real-valued functions on Z with the Tychonoff product topology (which corresponds to the point-wise convergence of the functions).

Our first theorem is quite general and concerns stratifiable spaces, see [7] for their definitions and properties. Here we mention only that each metrizable space is stratifiable, each stratifiable space is perfectly paracompact, and every subspace of a stratifiable space is stratifiable too.

Theorem 1. *Suppose Y is a stratifiable space and X is a closed subspace of Y with $|X| \geq 2$. There exists a positive linear extension operator $T : \mathbb{R}^{X \times X} \rightarrow \mathbb{R}^{Y \times Y}$ preserving constant functions, bounded functions, continuous functions, pseudometrics, and metrics. This operator is continuous with respect to each of the three topologies: point-wise convergence, uniform, and compact-open.*

Obviously the phrase “ T preserves bounded functions, etc.” means that T carries bounded functions, etc., on $X \times X$ into bounded functions, etc., on $Y \times Y$.

For metrizable spaces we are able to prove much more. It will be convenient to formulate the corresponding result in terms of uniform spaces (see Chapter 8 of [11] for the theory of uniform spaces). We remark that each metric space is automatically a uniform space. We call a uniform space *metrizable* if its uniformity is generated by a metric.

Theorem 2. *Suppose Y is a metrizable uniform space and X is a closed subspace of Y with $|X| \geq 2$. There exists a positive linear extension operator $T : \mathbb{R}^{X \times X} \rightarrow \mathbb{R}^{Y \times Y}$ preserving constant functions, bounded functions, continuous functions, pseudometrics, metrics, admissible metrics, dominating metrics, and uniformly dominating metrics. This operator is continuous with respect to each of the three topologies: point-wise convergence, uniform, and compact-open. Moreover, if the uniform space Y is complete, then T preserves complete continuous uniformly dominating metrics. If Y is totally bounded and $\dim(Y \setminus X) < \infty$, then T preserves totally bounded pseudometrics.*

A metric d on a topological [resp. uniform] space Z is called *dominating* [resp. *uniformly dominating*] if the formal identity map from the metric space (Z, d) to Z is [uniformly] continuous. A metric which is continuous and dominating is said to be *admissible*.

The proofs of the two theorems exploit Hartman-Mycielski space $\text{HM}(X)$ of all X -valued step functions defined on the interval $[0, 1)$ (in a similar way as Zarichnyi [16] applied the space of all X -valued measurable functions) and also Pikhurko’s [15] idea of constructing the required operator T as sum of a series of operators “separating” points of Y .

Theorems 1 and 2 will be applied to construct linear operators extending invariant metrics. For a topological space X by $C(X \times X)$ we denote the linear lattice of continuous functions on $X \times X$, equipped with the compact-open topology. If a compact topological group G acts on X , let

$$C_{\text{inv}}(X \times X) = \{f \in C(X \times X) : f(gx) = f(gy) \text{ for all } g \in G \text{ and } x, y \in X\}$$

denote the subspace of $C(X \times X)$ consisting of all G -invariant functions.

Theorem 3. *Suppose a compact topological group G acts on a stratifiable space Y , and X is a G -invariant subspace of Y with $|X| \geq 2$. There exists a positive linear continuous (in the compact-open topology) extension operator*

$T : C_{inv}(X \times X) \rightarrow C_{inv}(Y \times Y)$ preserving constant functions, bounded invariant functions, invariant pseudometrics, invariant metrics. If the space Y is metrizable, then additionally T preserves admissible metrics. If the group G is finite, then the operator T is continuous with respect to the point-wise convergence of functions.

The last theorem is an improvement obtained by the second author of a former result of [1] thanks to a discussion with C. Atkin. Another contribution of the second author is Section 5 containing a relatively simple construction of extension operators S , S_1 , S_2 having almost all properties of the operator T from Theorems 1 and 2 (except that S does not preserve metrics, S_1 fails to preserve constants, and S_2 is not positive).

1. HARTMAN-MYCIELSKI CONSTRUCTION

This construction appeared in [14] in connection with some problems of topological algebra, see also [9]. For an $n \in \mathbb{N}$ and a topological space X let $\text{HM}_n(X)$ be the set of all functions $f : [0, 1] \rightarrow X$ for which there exists a sequence $0 = a_0 < a_1 < \dots < a_n = 1$ such that f is constant on each interval $[a_{i-1}, a_i]$, $1 \leq i \leq n$. Let $\text{HM}(X) = \bigcup_{n \in \mathbb{N}} \text{HM}_n(X)$.

A neighborhood sub-base of the topology of $\text{HM}(X)$ at an $f \in \text{HM}(X)$ consists of sets $N(a, b, V, \varepsilon)$, where

- 1) $0 \leq a < b \leq 1$, f is constant on $[a, b]$, V is a neighborhood of $f(a)$ in X , and $\varepsilon > 0$;
- 2) $g \in N(a, b, V, \varepsilon)$ means that $|\{t \in [a, b] : g(t) \notin V\}| < \varepsilon$, where $|\cdot|$ denotes the Lebesgue measure.

As noted in [9, Proposition 2] for every subspace A of X , the space $\text{HM}(A)$ can be considered as a subspace of $\text{HM}(X)$. Also, the space X can be identified with the subspace $\text{HM}_1(X)$ of $\text{HM}(X)$.

For an element $f \in \text{HM}(X)$ let $\text{supp}(f)$ denote the smallest subset $A \subset X$ such that $f \in \text{HM}(A) \subset \text{HM}(X)$. Evidently that $\text{supp}(f) = f([0, 1])$.

Recall that for a space Z the space \mathbb{R}^Z is endowed with the Tychonoff product topology (which corresponds to the point-wise convergence on \mathbb{R}^Z considered as a function space).

Proposition 1. *The formula*

$$hm(d)(f, g) = \int_0^1 d(f(t), g(t)) dt$$

defines a positive linear continuous extension operator

$$hm : \mathbb{R}^{X \times X} \rightarrow \mathbb{R}^{\text{HM}(X) \times \text{HM}(X)}$$

preserving constant functions, bounded functions, and bounded continuous functions, pseudometrics, metrics, dominating metrics, and bounded admissible metrics. Moreover, for any totally bounded pseudometric d on X the pseudometric $hm(d)$ is totally bounded on each $\text{HM}_n(X)$, $n \in \mathbb{N}$.

Proof. It is an easy exercise to show that hm is a positive linear continuous extension operator preserving constant functions, bounded functions, and [pseudo]metrics. From Proposition 5 of [9] and its proof it follows that hm preserves dominating metrics and bounded admissible metrics.

Let us show that hm preserves bounded continuous functions. For this fix a bounded continuous function $d : X \times X \rightarrow \mathbb{R}$, $\varepsilon > 0$ and two elements $f, g \in \text{HM}(X)$. Without loss of generality, $|d(x, x')| \leq 1$ for every $x, x' \in X$. Let $0 = a_0 < a_1 < \dots < a_n = 1$ be a sequence such that both f and g are constant on each interval $[a_{i-1}, a_i]$, $1 \leq i \leq n$. Using the continuity of d , for every $i \in \{1, \dots, n\}$ pick neighborhoods $U_i, V_i \subset X$ of $f(a_i)$, $g(a_i)$, respectively, such that for every $x \in U_i$, $y \in V_i$ we have $|d(x, y) - d(f(a_i), g(a_i))| < \varepsilon/2$. Then $U = \bigcap_{i=1}^n N(a_{i-1}, a_i, U_i, \frac{\varepsilon}{8n})$ and $V = \bigcap_{i=1}^n N(a_{i-1}, a_i, V_i, \frac{\varepsilon}{8n})$ are neighborhoods of f and g , respectively, such that for every $f' \in U$, $g' \in V$ we have $|hm(d)(f', g') - hm(d)(f, g)| < \varepsilon$. That means the function $hm(d) : \text{HM}(X) \times \text{HM}(X) \rightarrow \mathbb{R}$ is continuous.

Finally, we show that for every totally bounded pseudometric d on X the pseudometric $hm(d)$ is totally bounded on each $\text{HM}_n(X)$. Fix $n \in \mathbb{N}$ and a totally bounded pseudometric d on X . Consider the equivalence relation \sim on X , where $x \sim y$ if $d(x, y) = 0$. Then the pseudometric d induces a totally bounded metric ρ on the quotient space X/\sim . Let $(\tilde{X}, \tilde{\rho})$ denote the completion of X/\sim by the metric ρ and let $p : X \rightarrow X/\sim \subset \tilde{X}$ be the quotient map. Clearly, the space \tilde{X} is compact. Then the space $\text{HM}_n(\tilde{X})$ is compact as a continuous image of the product $\Delta^{n-1} \times \tilde{X}^n$, where $\Delta^{n-1} = \{(a_0, \dots, a_n) : 0 = a_0 \leq a_1 \leq \dots \leq a_n = 1\}$ is an $(n-1)$ -dimensional simplex, see [9, p.217]. The metric $hm(\tilde{\rho})$, being continuous, is totally bounded on $\text{HM}_n(\tilde{X})$. Since for every $f, g \in \text{HM}_n(X)$ $p \circ f, p \circ g \in \text{HM}_n(\tilde{X})$ and $hm(d)(f, g) = hm(\tilde{\rho})(p \circ f, p \circ g)$, we get $hm(d)$ is a totally bounded pseudometric on $\text{HM}_n(X)$. \square \square

For a space X by $\exp_\omega X$ we denote the set of all finite subsets of X . A map $u : Y \rightarrow \exp_\omega X$ is called *upper-semicontinuous* provided for every open set $U \subset X$ the set $\{y \in Y \mid u(y) \subset U\}$ is open in Y .

Next, we prove that the spaces $\text{HM}(X)$ over stratifiable spaces have an important extension property. Below for a metric d on a space X the open d -ball $\{x' \in X : d(x', x) < \varepsilon\}$ of radius ε around a point $x \in X$ is denoted by $O_d(x, \varepsilon)$.

Proposition 2. *For every closed subset X of a stratifiable space Y there exist*

- 1) *an upper semi-continuous map $u : Y \rightarrow \exp_\omega X$ such that $u(x) = \{x\}$ and*

- 2) a continuous map $h : Y \rightarrow \mathbf{HM}(X)$ extending the identity embedding $X \hookrightarrow \mathbf{HM}(X)$ such that $\text{supp}(h(y)) \subset u(y)$ for every $y \in Y$.

Moreover, if $\dim(Y \setminus X) < n$, then $h(Y) \subset \mathbf{HM}_n(X)$. If d is an admissible metric for Y , then the map u can be chosen so that $u(y) \subset O_d(y, 2d(y, X))$ for every $y \in Y$.

Proof. Suppose X is a closed subset of a stratifiable space Y . By the proof of Theorem 4.3 of [7], there exists a locally finite open cover \mathcal{U} of $Y \setminus X$ and a map $\alpha : \mathcal{U} \rightarrow X$ such that the map $u : Y \rightarrow \exp_\omega(X)$ defined by $u(y) = \{y\}$ for $y \in X$ and $u(y) = \{\alpha(U) \mid y \in \text{cl}(U), U \in \mathcal{U}\}$ for $y \in Y \setminus X$ is upper semi-continuous. Let \leq be any linear ordering of the set \mathcal{U} and let $\{\lambda_U : Y \setminus X \rightarrow [0, 1]\}_{U \in \mathcal{U}}$ be a partition of unity subordinate to the cover \mathcal{U} . For a $y \in Y \setminus X$ define a function $h(y) \in \mathbf{HM}(X)$ letting

$$h(y)(t) = \alpha(U), \quad \text{if} \quad \sum_{V < U} \lambda_V(y) \leq t < \sum_{V \leq U} \lambda_V(y).$$

Because only finitely many of $\lambda_V(y)$'s are distinct from zero, the function $h(y)$ is well-defined.

For $y \in X$ let $h(y) = y \in X \subset \mathbf{HM}(X)$.

We claim that the so-defined map $h : Y \rightarrow \mathbf{HM}(X)$ is continuous and satisfies the requirements of Proposition 2. The inclusion $\text{supp}(h(y)) \subset u(y)$, $y \in Y$, follows from the definitions of $h(y)$ and $u(y)$.

The continuity of h on the set $Y \setminus X$ easily follows from the local finiteness of the cover \mathcal{U} . Let us verify the continuity of h at a point $x \in X$. Fix any neighborhood U of $h(x) = x$ in $\mathbf{HM}(X)$. According to the definition of the topology of $\mathbf{HM}(X)$, there exists a neighborhood V of x in X such that $\mathbf{HM}(V) \subset U$. Since the map $u : Y \rightarrow \exp_\omega X$ is upper-semicontinuous and $u(x) = \{x\}$, there is a neighborhood W of x in Y such that $u(y) \subset V$ for every $y \in W$. Then for such y we have $h(y) \in \mathbf{HM}(u(y)) \subset \mathbf{HM}(V) \subset U$, i.e. h is continuous at the point x .

If $\dim(Y \setminus X) < n$ then the cover \mathcal{U} can be chosen to be of order $\leq n$. In this case, according to the construction, we get $h(Y) \subset \mathbf{HM}_n(X)$.

If Y is a metrizable space with an admissible metric d , then using the classical technique of Dugundji [10] we may construct the map u so that $u(y) \subset O_d(y, 2d(y, X))$ for every $y \in Y$. \square

Question 1. Is $\mathbf{HM}(X)$ an absolute extensor for stratifiable spaces? The answer is “yes” for separable metrizable X . (This can be shown applying the arguments of [6, Ch.VI, §7]).

2. CONSTRUCTION OF AN EXTENSION OPERATOR T

Suppose X is a closed subset of a stratifiable space Y and a, b be two distinct points of X . An operator T satisfying the requirements of Theorems 1 and 2 will be constructed as the sum of a series $\sum_{n=1}^{\infty} 2^{-n} T_n$, where the collection of extension operators $\{T_n : \mathbb{R}^{X \times X} \rightarrow \mathbb{R}^{Y \times Y}\}_{n=1}^{\infty}$ “separates” points of Y .

It is known that every stratifiable space admits a bijective continuous map onto a metrizable space (combine [Bo, Lemma 8.2] with [Bo, property (A) on p.2]). Therefore, there is a continuous metric $d \leq 1$ on Y . Moreover, applying Theorem 5.2 of [7], we may adjust the metric d so that $d(y, X) > 0$ for every $y \in Y \setminus X$, where, as usual, $d(y, X) = \inf\{d(y, x) : x \in X\}$. If Y is a metrizable uniform space, then d will be assumed to generate the uniformity of Y .

Let $h : Y \rightarrow \mathbf{HM}(X)$ and $u : Y \rightarrow \exp_\omega(X)$ be the maps from Proposition 2 (in case $\dim Y \setminus X < \infty$ we assume that $h(Y) \subset \mathbf{HM}_k(X)$ for some $k \in \mathbb{N}$).

For every $n \in \mathbb{N}$ we shall define an extension operator $T_n : \mathbb{R}^{X \times X} \rightarrow \mathbb{R}^{Y \times Y}$ as follows. Fix $n \in \mathbb{N}$. Let \mathcal{U}_n be a locally finite (resp. finite, if the metric d is totally bounded) open cover of the space Y such that $\text{diam}_d(U) < 2^{-n}$ for every $U \in \mathcal{U}_n$, and let $\{\lambda_U^n : Y \rightarrow [0, 1]\}_{U \in \mathcal{U}_n}$ be a partition of unity, subordinate to the cover \mathcal{U}_n . Further we consider \mathcal{U}_n as a discrete topological space. Let \leq be any linear ordering on \mathcal{U}_n and let $h_n : Y \rightarrow \mathbf{HM}(\mathcal{U}_n)$ be the map defined for a $y \in Y$ by the formula

$$h_n(y)(t) = U, \quad \text{if} \quad \sum_{V < U} \lambda_V^n(y) \leq t < \sum_{V \leq U} \lambda_V^n(y).$$

As in the proof of Proposition 2, it can be shown that the map h_n is continuous.

By $X \sqcup \mathcal{U}$ denote the disjoint union of the spaces X and \mathcal{U}_n , $n \in \mathbb{N}$. According to [9, Proposition 2], we may identify $\mathbf{HM}(X)$ and $\mathbf{HM}(\mathcal{U}_n)$ with subspaces of $\mathbf{HM}(X \sqcup \mathcal{U})$. Finally, define a map $f_n : Y \rightarrow \mathbf{HM}(X \sqcup \mathcal{U})$ letting for a $y \in Y$

$$f_n(y)(t) = \begin{cases} h_n(y)(t), & \text{if } 0 \leq t < \min\{1, n d(y, X)\}; \\ h(y)(t), & \text{if } \min\{1, n d(y, X)\} \leq t < 1. \end{cases}$$

It is easily seen that f_n is a continuous map extending the natural embedding $X \subset \mathbf{HM}(X) \subset \mathbf{HM}(X \sqcup \mathcal{U})$.

Let us consider the linear operator $E : \mathbb{R}^{X \times X} \rightarrow \mathbb{R}^{(X \sqcup \mathcal{U}) \times (X \sqcup \mathcal{U})}$ defined for every $p \in \mathbb{R}^{X \times X}$ by

$$E(p)(x, y) = \begin{cases} p(x, y), & \text{if } x, y \in X; \\ \frac{1}{2}p(x, a) + \frac{1}{2}p(x, b), & \text{if } x \in X, y \in \mathcal{U}; \\ \frac{1}{2}p(a, y) + \frac{1}{2}p(b, y), & \text{if } x \in \mathcal{U}, y \in X; \\ p(a, b), & \text{if } x, y \in \mathcal{U} \text{ and } x \neq y; \\ 0, & \text{if } x = y; \end{cases}$$

(recall that a, b are two fixed point in X). One can easily verify that E is a positive linear continuous extension operator preserving constants, bounded, bounded continuous functions and [pseudo]metrics.

The operator $T_n : \mathbb{R}^{X \times X} \rightarrow \mathbb{R}^{Y \times Y}$ is defined as the composition

$$\mathbb{R}^{X \times X} \xrightarrow{E} \mathbb{R}^{(X \sqcup \mathcal{U}) \times (X \sqcup \mathcal{U})} \xrightarrow{hm} \mathbb{R}^{HM(X \sqcup \mathcal{U}) \times HM(X \sqcup \mathcal{U})} \xrightarrow{(f_n \times f_n)^*} \mathbb{R}^{Y \times Y},$$

where $(f_n \times f_n)^*(p) = p \circ (f_n \times f_n)$ for $p \in \mathbb{R}^{HM(X \sqcup \mathcal{U}) \times HM(X \sqcup \mathcal{U})}$, equivalently, by the explicit formula

$$T_n(p)(y, y') = \int_0^1 E(p)(f_n(y)(t), f_n(y')(t)) dt \quad \text{for } p \in \mathbb{R}^{X \times X}, \quad y, y' \in Y.$$

Remark that T_n is a positive linear continuous extension operator preserving constants, bounded, bounded continuous functions and pseudometrics.

Finally, we define the required operator $T : \mathbb{R}^{X \times X} \rightarrow \mathbb{R}^{Y \times Y}$ by the formula

$$T = \sum_{n=1}^{\infty} \frac{1}{2^n} T_n.$$

We shall verify the properties of the operator T . First, observe that the definition of T is correct, i.e. for every function $p : X \times X \rightarrow \mathbb{R}$ and every $y, y' \in Y$ the series $\sum_{n=1}^{\infty} 2^{-n} T_n(p)(y, y')$ is convergent. This is trivial, when $y, y' \in X$ (all T_n 's are extension operators). If $y \in X$ and $y' \notin X$ then for every $n \in \mathbb{N}$ with $d(y', X) \geq \frac{1}{n}$, by the construction of T_n , we have $T_n(p)(y, y') = \frac{1}{2}p(y, a) + \frac{1}{2}p(y, b)$. If $y, y' \notin X$ then, for every $n \in \mathbb{N}$ with $d(y, X), d(y', X) \geq \frac{1}{n}$, we have $|T_n(p)(y, y')| \leq |p(a, b)|$. These remarks imply that the series $\sum_{n=1}^{\infty} 2^{-n} T_n(y, y')$ converges for every $y, y' \in Y$, i.e. the definition of T is correct.

Since T_n 's are positive linear extension operators preserving constants, bounded functions, bounded continuous functions and pseudometrics, so is the operator T .

3. PROOF OF THEOREM 1

In an obvious way Theorem 1 follows from the above-mentioned properties of the operator T and the subsequent four lemmas. The first of them can be easily derived from the construction of T .

Lemma 1. *Let $y, y' \in Y$ and $A = \text{supp}(h(y)) \cup \text{supp}(h(y')) \cup \{a, b\}$. If $p, p' : X \times X \rightarrow \mathbb{R}$ satisfy $p|_A \times A \leq p'|_A \times A$, then $T(p)(y, y') \leq T(p')(y, y')$. Moreover, if $p|_A \times A \equiv c$, then $T(p)(y, y') = c$.*

Lemma 2. *The operator $T : \mathbb{R}^{X \times X} \rightarrow \mathbb{R}^{Y \times Y}$ is continuous with respect to the uniform, point-wise or compact-open topologies on the function spaces $\mathbb{R}^{X \times X}$ and $\mathbb{R}^{Y \times Y}$.*

Proof. Because the operator T is positive and preserves constant functions, it is continuous with respect to the uniform convergence of functions.

Let us show that the operator T is continuous with respect to the point-wise convergence of functions. For this, fix points $y, y' \in Y$ and notice that the set $A = \{a, b\} \cup \text{supp}(h(y)) \cup \text{supp}(h(y'))$ is finite. By Lemma 1, for a function $p : X \times X \rightarrow \mathbb{R}$ the inequality $|p(x, x')| \leq 1$ for every $(x, x') \in A \times A$ implies $|T(p)(y, y')| \leq 1$. This means that the operator T is continuous with respect to the point-wise convergence of functions.

To show that T is continuous with respect to the compact-open topology fix a compactum $C \subset Y \times Y$ and notice that the set $K' = \bigcup \{u(y) \mid y \in \text{pr}_1(C) \cup \text{pr}_2(C)\} \subset X$ is compact because of the upper-semicontinuity of the map $u : Y \rightarrow \exp_{\omega} X$ (see [12, Theorem VI.7.10]) (by $\text{pr}_i : Y \times Y \rightarrow Y$ we denote the projection onto the corresponding factor). Consider the compact set $K = K' \cup \{a, b\}$. Then $\text{supp}(h(y)) \cup \text{supp}(h(y')) \subset u(y) \cup u(y') \subset K$ for every $(y, y') \in C$. Now Lemma 1 yields that for a function $p : X \times X \rightarrow \mathbb{R}$ if $|p(x, x')| \leq 1$ for every $(x, x') \in K \times K$ then $|T(p)(y, y')| \leq 1$ for every $(y, y') \in C$. But this means that the operator T is continuous in the compact-open topology. \square

Lemma 3. *The operator T preserves continuous functions.*

Proof. Let $p : X \times X \rightarrow \mathbb{R}$ be a continuous function. Fix any point $(y_0, y'_0) \in Y \times Y$. Let $M = \max\{|p(x, x')| : x, x' \in \{a, b\} \cup u(y_0) \cup u(y'_0)\}$. Since the map p is continuous, there is a neighborhood $U \subset X$ of the compactum $\{a, b\} \cup u(y_0) \cup u(y'_0)$ such that $|p(x, x')| < M + 1$ for every $x, x' \in U$. Since the map $u : Y \rightarrow \exp_\omega X$ is upper-semicontinuous, there are neighborhoods $V, V' \subset Y$ of y_0, y'_0 respectively such that for every $y \in V$ and $y' \in V'$ we have $u(y) \cup u(y') \subset U$.

Now consider the bounded continuous function $\tilde{p} : X \times X \rightarrow \mathbb{R}$ defined by the formula

$$\tilde{p}(x, x') = \begin{cases} p(x, x'), & \text{if } -M - 1 \leq p(x, x') \leq M + 1 \\ M + 1, & \text{if } p(x, x') \geq M + 1 \\ -M - 1, & \text{if } p(x, x') \leq -M - 1. \end{cases}$$

Obviously that $\tilde{p}|_U = p|_U$. Moreover, since the operator T preserves bounded continuous functions, the map $T(\tilde{p}) : Y \times Y \rightarrow \mathbb{R}$ is continuous. Now remark that for every $(y, y') \in V \times V'$ $\text{supp}(h(y)) \cup \text{supp}(h(y')) \subset \{a, b\} \cup u(y) \cup u(y') \subset U$. Since $\tilde{p}|_U = p|_U$, by Lemma 1, $T(p)(y, y') = T(\tilde{p})(y, y')$. Therefore, $T(p)|_{V \times V'} = T(\tilde{p})|_{V \times V'}$ and the function $T(p)$ is continuous. \square

Lemma 4. *The operator T preserves metrics.*

Proof. Let p be a metric on X . Since the operator T preserves pseudometrics, it remains to prove that $T(p)(y, y') \neq 0$ for distinct $y, y' \in Y$. So, fix $y, y' \in Y$ with $y \neq y'$.

If $y, y' \in X$ then $T(p)(y, y') = p(y, y') \neq 0$ because p is a metric on X . Now assume that $y \in X$ and $y' \notin X$. Then $d(y', X) > \frac{1}{n}$ for some $n \in \mathbb{N}$. Consequently, $f_n(y) = y \in X \subset \text{HM}(X \sqcup \mathcal{U})$ and $f_n(y') = h_n(y') \in \text{HM}(\mathcal{U}_n) \subset \text{HM}(X \sqcup \mathcal{U})$. By the property of the operator E , we have $E(p)(y, h_n(y')(t)) = \frac{1}{2}(p(y, a) + p(y, b)) \geq \frac{1}{2}p(a, b)$ for every $t \in [0, 1]$ and thus

$$T_n(p)(y, y') = \int_0^1 E(p)(y, h_n(y')(t)) dt \geq \frac{1}{2}p(a, b) > 0.$$

This yields $T(p)(y, y') \geq 2^{-n}T_n(p)(y, y') > 0$.

Now assume that $y, y' \in Y \setminus X$. Then there is an $n \in \mathbb{N}$ such that $d(y, X) > n^{-1}$, $d(y', X) > n^{-1}$ and $d(y, y') > 2^{-n+1}$. In this case, $f_n(y) = h_n(y)$ and $f_n(y') = h_n(y')$. Since $\text{diam}(U) < 2^{-n}$ for $U \in \mathcal{U}_n$, there is no $U \in \mathcal{U}_n$ with $\{y, y'\} \subset U$. Consequently, $\text{supp}(h_n(y)) \cap \text{supp}(h_n(y')) = \emptyset$. By the definition of the metric $E(p)$, $E(p)(h_n(y)(t), h_n(y')(t)) = p(a, b)$ for every $t \in [0, 1]$. Then

$$2^n T(p)(y, y') \geq T_n(p)(y, y') = \int_0^1 E(p)(h_n(y)(t), h_n(y')(t)) dt = p(a, b) > 0$$

Therefore, $T(p)$ is a metric on Y . \square

4. PROOF OF THEOREM 2

In this section we suppose that Y is a metrizable uniform space and the metric d generates the uniformity of Y . Moreover, the map u constructed in Proposition 2 has the following property: $u(y) \subset O_d(y, 2d(y, X))$ for every $y \in Y$.

In an obvious way Theorem 2 follows from Theorem 1 and the subsequent four lemmas.

Lemma 5. *The operator T preserves the class of dominating metrics.*

Proof. Let p be a dominating metric for X . To show that the metric $T(p)$ dominates the topology of Y , it suffices for every $y \in Y$ and every $\varepsilon \in (0, 1]$ to find $\delta > 0$ such that $T(p)(y, y') \geq \delta$ for every $y' \in Y$ with $d(y', y) > \varepsilon$.

First we consider the case $y \notin X$. Then we can find $n \in \mathbb{N}$ such that $d(y, X) > \frac{1}{n}$ and $2^{-n+1} < \varepsilon$. Let $\delta = 2^{-n-1}p(a, b)$ and $y' \in Y$ be any point with $d(y, y') > \varepsilon$. Then $d(y, y') > 2^{-n+1}$ and by the choice of the cover \mathcal{U}_n and the map h_n , we have $\text{supp}(h_n(y)) \cap \text{supp}(h_n(y')) = \emptyset$. As we have observed in the proof of Lemma 4, this implies $E(p)(h_n(y)(t), h_n(y')(t)) = p(a, b)$ for every $t \in [0, 1]$. Besides, it follows that $E(p)(h_n(y)(t), h(y')(t)) \geq \frac{1}{2}p(a, b)$. Then

$$2^n T(p)(y, y') \geq T_n(p)(y, y') = \int_0^1 E(p)(h_n(y)(t), f_n(y')(t)) dt \geq \frac{1}{2}p(a, b) = 2^n \delta.$$

Now assume that $y \in X$. Let $n \in \mathbb{N}$ be such that $2^{-n+1} < \varepsilon$. Since the metric p is dominating for X , there is $\eta > 0$ such that $p(y, x) > \eta$ for every $x \in X$ with $d(y, x) > \varepsilon$. Let $\delta = \min\{2^{-n-1}p(a, b), n\varepsilon 2^{-n-3}p(a, b), 3\eta/8\}$ and fix any point $y' \in Y$ with $d(y, y') > \varepsilon$. To verify that $T(p)(y, y') \geq \delta$, consider two cases:

1) $d(y', X) \geq \frac{\varepsilon}{4}$. By the property of the metric $E(p)$, we have $E(p)(y, h_n(y')(t)) \geq \frac{1}{2}p(a, b)$ for every $t \in [0, 1]$. Then

$$\begin{aligned} 2^n T(p)(y, y') &\geq T_n(p)(y, y') = \int_0^1 E(p)(y, f_n(y')(t)) dt \geq \int_0^{\min\{1, d(y', X)\}} E(p)(y, h_n(y')(t)) dt \geq \\ &\geq \min\{1, n d(y', X)\} \frac{1}{2}p(a, b) > 2^{-1} \min\{1, n\varepsilon/4\}p(a, b) \geq 2^n \delta. \end{aligned}$$

Now pass to the other case:

2) $d(y', X) < \frac{\varepsilon}{4}$. Then $\text{supp}(h(y')) \subset u(y') \subset O_d(y', \frac{\varepsilon}{2})$ and because $d(y, y') > \varepsilon$, we get $d(y, h(y')(t)) > \frac{\varepsilon}{2}$ for every $t \in [0, 1)$. By the choice of η , this implies $p(y, h(y')(t)) > \eta$ for every $t \in [0, 1)$. Then

$$\begin{aligned} 2T(p)(y, y') &\geq T_1(p)(y, y') = \int_0^1 E(p)(y, f_1(y')(t)) dt \geq \\ &\geq \int_{d(y', X)}^1 E(p)(y, h(y')(t)) dt \geq \int_{\varepsilon/4}^1 p(y, h(y')(t)) dt \geq (1 - \frac{\varepsilon}{4})\eta \geq \frac{3}{4} \cdot \eta \geq 2\delta. \end{aligned}$$

□

Lemma 6. *The operator T preserves uniformly dominating metrics.*

Proof. Let p be a uniformly dominating metric for the uniform space X . To show that the metric $T(p)$ is uniformly dominating for Y , it suffices to verify that the formal identity map $(Y, T(p)) \rightarrow (Y, d)$ between the respective metric spaces is uniformly continuous.

Fix any $\varepsilon > 0$. We have to find $\delta > 0$ such that for every $y_1, y_2 \in Y$ the inequality $T(p)(y_1, y_2) < \delta$ implies $d(y_1, y_2) < \varepsilon$; equivalently, $d(y_1, y_2) \geq \varepsilon$ implies $T(p)(y_1, y_2) \geq \delta$. To find such δ , select $n \in \mathbb{N}$ so that $2^{-n} < \frac{\varepsilon}{2}$ and $\frac{n}{2^{n+2}} < \frac{1}{2}$. Since the metric p is uniformly dominating for X , there exists $\delta > 0$ such that $d(x, x') < \frac{\varepsilon}{2}$ for all $x, x' \in X$ with $p(x, x') < 2^{n+1}\delta$. Moreover, we may take δ so small that $2^n\delta < \frac{n}{2^{n+3}}p(a, b)$.

We claim that the so-chosen number δ satisfies our requirements. To show this, fix any points $y_1, y_2 \in Y$ with $d(y_1, y_2) \geq \varepsilon$. Because $T(p)(y_1, y_2) \geq 2^{-n}T_n(y_1, y_2)$, it suffices to verify the inequality $T_n(y_1, y_2) \geq 2^n\delta$. Two cases will be considered separately:

1) $\max\{d(y_1, X), d(y_2, X)\} \geq 2^{-n-2}$. Without loss of generality, $d(y_1, X) \leq d(y_2, X)$. Since $d(y_1, y_2) \geq \varepsilon$ and $\sup_{U \in \mathcal{U}_n} \text{diam}(U) < 2^{-n} < \frac{\varepsilon}{2}$, we get $\text{supp}(h_n(y_1)) \cap \text{supp}(h_n(y_2)) = \emptyset$. It follows that $E(p)(h_n(y_1)(t), h_n(y_2)(t)) \geq p(a, b)$ and $E(p)(h_n(y_2), h(y_1)) \geq \frac{1}{2}p(a, b)$ for every $t \in [0, 1)$. Then

$$\begin{aligned} T_n(p)(y_1, y_2) &= \int_0^1 E(p)(f_n(y_1)(t), f_n(y_2)(t)) dt \geq \\ &\geq \frac{1}{2} \min\{1, nd(y_2, X)\}p(a, b) \geq \frac{1}{2}p(a, b)\frac{n}{2^{n+2}} > 2^n\delta. \end{aligned}$$

Next, we consider the case:

2) $\max\{d(y_1, X), d(y_2, X)\} < 2^{-n-2}$. By the definition of the map u , we have $\text{supp}(h(y_i)) \subset u(y_i) \subset O_d(y_i, 2d(y_i, X)) \subset O_d(y_i, \frac{\varepsilon}{4})$ for $i = 1, 2$. Consequently,

$$d(h(y_1)(t), h(y_2)(t)) > \frac{\varepsilon}{2} \text{ for every } t \in [0, 1)$$

and by the choice of δ , we get $p(h(y_1)(t), h(y_2)(t)) > 2^{n+1}\delta$. Finally, for the pseudometric $T_n(p)$ we obtain

$$\begin{aligned} T_n(p)(y_1, y_2) &= \int_0^1 E(p)(f_n(y_1)(t), f_n(y_2)(t)) dt \geq \int_{1-n \max\{d(y_1, X), d(y_2, X)\}}^1 p(h(y_1)(t), h(y_2)(t)) dt \geq \\ &\geq (1 - n \max\{d(y_1, X), d(y_2, X)\})2^{n+1}\delta \geq (1 - n2^{-n-2})2^{n+1}\delta \geq \frac{1}{2}2^{n+1}\delta = 2^n\delta. \end{aligned}$$

□

Lemma 7. *If the uniform space Y is complete, then the operator T preserves complete continuous uniformly dominating metrics.*

Proof. Observe that in a complete metrizable uniform space every continuous uniformly dominating metric is complete and apply Lemmas 3 and 6. □

Lemma 8. *If the uniform space Y is totally bounded and $\dim Y \setminus X < \infty$ then the operator T preserves totally bounded pseudometrics.*

Proof. Fix a totally bounded pseudometric p on X . It is enough to show that each pseudometric $T_n(p)$ is totally bounded. Fix any $n \in \mathbb{N}$. Since the metric d on Y is totally bounded, by the construction, the cover \mathcal{U}_n is finite. Then the metric $E(p)$ on $X \sqcup \mathcal{U}_n$ is totally bounded. Let $k > |\mathcal{U}_n|$ be such that $h(Y) \subset \text{HM}_k(X)$. Then $f_n(Y) \subset \text{HM}_{2k}(X \sqcup \mathcal{U}_n)$ and $T_n(p)(y, y') = hm(E(p))(f_n(y), f_n(y'))$ for every $y, y' \in Y$. By Proposition 1, the pseudometric $hm(E(p))$ is totally bounded on $\text{HM}_{2k}(X \sqcup \mathcal{U}_n)$. Hence, the pseudometric $T_n(p)$ is totally bounded on Y . □

5. PROOF OF THEOREM 3

Assume that G is a compact group, μ the Haar measure on G , Y is a (left) G -space and X is a closed subspace of Y consisting of at least two points and invariant under the action of G . For $Z \in \{X, Y\}$ by $C(Z \times Z)$ we denote the linear lattice of continuous functions on $Z \times Z$, equipped with the compact-open topology and by $C_{inv}(Z \times Z)$ its linear subspace consisting of continuous *invariant* functions, i.e., such that $f(gx, gy) = f(x, y)$ for every $g \in G$ and $x, y \in X$.

Proposition 3. *The averaging operator $A : C(Y \times Y) \rightarrow C_{inv}(Y \times Y)$ defined by*

$$Af(y, y') = \int_G f(gy, gy') d\mu \quad \text{for } f \in C(Y \times Y), \quad y, y' \in Y$$

is a continuous retraction of $C(Y \times Y)$ onto $C_{inv}(Y \times Y)$. The operator A takes constants, pseudometrics, metrics, admissible metrics into constants, invariant pseudometrics, invariant metrics, invariant admissible metrics, respectively.

Proof. Let $d \in C(Y \times Y)$ be a metric on Y and $d' = Ad$. Let $a, b \in Y$, $a \neq b$. There is a neighborhood U of the neutral element of the group G such that $d(ga, gb) \geq 2^{-1}d(a, b)$ for $g \in U$. Therefore $d'(a, b) \geq 2^{-1}\mu(U)d(a, b) > 0$, i.e., $d' = Ad$ is a metric.

Now assume that $d \in C(Y \times Y)$ is an admissible metric, and (y_n) is a sequence of points of Y such that $\lim_n d'(y_n, y) = 0$ for some $y \in Y$. Hence the sequence of real functions $\varphi_n(g) = d(gy_n, gy)$ tends to zero in the L_1 -norm, and since $\mu(G) = 1 < \infty$, there is a subsequence φ_{k_n} which tends to zero almost everywhere, in particular, $\lim_n d(g_0 y_{k_n}, g_0 y) = 0$ for some $g_0 \in G$. “Multiplying the last relation from the left” by g_0^{-1} we get $\lim_n d(y_{k_n}, y) = 0$. The same arguments yield that every subsequence of the sequence (y_n) contains a subsequence convergent (in the admissible metric d) to y . That means that the whole sequence (y_n) tends to y . We have proved that the $d' = Ad$ is dominating, and (being continuous) is admissible. The other assertions of the proposition are evident. \square

Proof of Theorem 3. Let T be the operator appearing in Theorem 1 (Theorem 2 in case of metrizable Y). The required operator I is the composition $I = A \circ T|_{C_{inv}(X \times X)}$. \square

6. THE EXTENSION OPERATORS $S, S_1, S_2 : \mathbb{R}^{X \times X} \rightarrow \mathbb{R}^{Y \times Y}$

In this section we present a simple construction of extension operators S, S_1, S_2 having almost all properties of the operator T .

Suppose Y is a stratifiable space, X is a closed subset of Y and a, b are two distinct points in X . As we said, the space Y admits a continuous metric $d \leq 1$ such that $d(y, X) > 0$ for all $y \in Y \setminus X$. If Y is metrizable, we assume that d is an admissible metric for Y .

For $y, y' \in Y$ let

$$d^*(y, y') = \min[d(y, y'), d(y, X) + d(y', X)],$$

Clearly, d^* is a continuous pseudometric on Y (moreover, the restriction of d^* on $Y \setminus X$ is a metric). Let $h : Y \rightarrow \text{HM}(X)$ be the map appearing in Proposition 2 and define

$$S(p)(y, y') = hm(p)(h(y), h(y')) = \int_0^1 p(h(y)(t), h(y')(t)) dt,$$

$$S_1(p) = S(p) + p(a, b)d^*, \quad S_2(p) = S(p) + (p(a, b) - p(a, a))d^*$$

for $p \in \mathbb{R}^{X \times X}$, $y, y' \in Y$. Thus we have defined three extension operators $S, S_1, S_2 : \mathbb{R}^{X \times X} \rightarrow \mathbb{R}^{Y \times Y}$.

Theorem 4. *The operators S, S_1 and S_2 satisfy the requirements of Theorem 1 except that S does not preserve metrics, S_1 fails to preserve constants and S_2 is not positive. Moreover, if the space Y is metrizable, then the operators S_1 and S_2 preserve dominating and admissible metrics.*

Proof. The first statement of the theorem easily follows from Propositions 1 and 2 (to prove that these operators preserve continuous functions one should apply the arguments from Lemma 3). The fact that in the metric case, S_1 and S_2 preserve dominating metrics is an immediate consequence of the next two easy lemmas.

Lemma 9. *For every dominating metric p on X the pseudometric $\rho = S(p)$ has the following property :*

(*) *Let $y_n \in Y$ for $n \in \mathbb{N}$ and $x \in X$. Then $\lim_n \rho(y_n, x) = 0$ and $\lim_n d(y_n, X) = 0$ imply $\lim_n d(y_n, x) = 0$.*

Proof. (cf. proof of Lemma 5). Recall that d is a fixed admissible metric for Y . According to the last assertion of Proposition 2 and the definition of the operator S , for every $y \in Y$ there is an $y' \in u(y) \subset X$ such that

$$d(y, y') \leq 2d(y, x) \quad \text{and} \quad p(y', x) \leq \rho(y, x).$$

We have $d(y_n, x) \leq d(y_n, y'_n) + d(y'_n, x) \leq 2d(y_n, X) + d(y'_n, x)$.

But $0 \leq p(y'_n, x) \leq \rho(y_n, x) \rightarrow 0$ as $n \rightarrow \infty$, and since p is a dominating metric for X , we get $\lim_n d(y'_n, x) = 0$, and by the assumption of (*), $\lim_n d(y_n, x) = 0$. \square

Lemma 10. *For every pseudometric ρ in Y and every constant $c > 0$ the sum $\rho + cd^*$ is a dominating metric on Y , provided ρ has the property $(*)$.*

Proof. By $(*)$, the sum $\rho + cd^*$ is dominating “at each point” $x \in X$. In order to show the domination at the remaining points it is enough to examine the second term d^* which is a metric, when restricted to $Y \setminus X$. \square

\square

Finally, we pose an open problem suggested by Theorem 2 and a known result of J.S. Isbell [13] according to which for every subspace X of a uniform space Y , every bounded uniformly continuous pseudometric on X extends to a bounded uniformly continuous pseudometric on Y .

Problem 1. Suppose X is a subspace of a metrizable uniform space Y . Does there exist a “nice” operator extending bounded uniformly continuous pseudometrics from X over Y .

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